Stability and Excess Noise in the MOLLER/P2 Integrating Detector Signal

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Abstract

This note discusses the theoretically expected signal variance from various noise sources in the integrating detector chain. It gives the relevant expressions that will serve as a basis from which to design detector PMT (photomultiplier) voltage dividers that are optimized with respect to noise, linearity, and long term stability. Noise sources are discussed primarily for the detector and the PMT. The properties of the front-end electronics and the associated noise bandwidth are discussed in a separate note, dealing only with the front-end electronics design. Both MOLLER and P2 seek to suppress excess noise to the percent level and I show that excess noise can significantly increase due to suboptimal detector design (detector resolution) as well as dynode gain fluctuations, if care is not taken in the design of the voltage divider.

1 Introduction

In this document I consider a source of noise any part of the detector signal that introduces a variance, even if that variance is a natural part of the operating condition of the equipment. The fundamental and irreducible (from the point of view of the detector) source of noise is associated with the variance in the event rate itself, in a given detector. The origin of this source is not discussed in this document, however. It is meant only to provide a basis, with respect to which the other sources of noise are compared. All the other sources of noise are considered excess noise above this fundamental source, which we refer to as counting statistics and which varies as a function of the number of detected events.

Contrary to counting mode experiments, where one can really talk about counting statistics because one literally counts individual pulses in a detector, in integration mode or current mode, counting statistics only manifests itself as shot noise at the photomultiplier cathode (in our case), which contributes to the root-mean-square (RMS) in a continuous signal (no pulse counting) [1].

Without considering variations in the event rate or the number of photons hitting the cathode per event (I'll come back to this below), the variance in the detector signal can then be quantified, to first order, simply as shot noise at the cathode originating only from the detector events of interest [1],

$$\sigma^2_{I_k} = 2qI_kB \ . \tag{1.1}$$

Where $q = \langle n \rangle e^-$ is the charge quantum per event, with $\langle n \rangle$ being the mean number of photoelectrons per detector event, $I_k = R\langle n \rangle e^-$ being the average cathode current, with and R being the detector event rate, and B is the integration bandwidth.

Neglecting any PMT related or electronic noise, integration factors, etc., the error in the measured asymmetry,

$$A = \frac{I_k^+ - I_k^-}{I_k^+ + I_k^-} , \tag{1.2}$$

based only on event statistics generated shot noise, would be given by

$$(\delta A)^2 = \left(\frac{\partial A}{\partial I_k^+} \right)^2 \sigma^2_{I_{k+}} + \left(\frac{\partial A}{\partial I_k^-} \right)^2 \sigma^2_{I_{k-}} = \frac{\sigma^2_{I_k}}{I_k^2} . \tag{1.3}$$

Where I used $I_{k^-} \simeq I_{k^+} \equiv I_k$ and the assumption has been made that $\sigma^2_{I_{k-}} = \sigma^2_{I_{k+}} \equiv \sigma^2_{I_k}$. From eqn. 1.1, one then finds

$$(\delta A)^2 = \frac{\sigma^2_{I_k}}{I_k^2} = \frac{2(\langle n \rangle e^-)(R\langle n \rangle e^-)B}{(R\langle n \rangle e^-)^2} = \frac{2B}{R} . \tag{1.4}$$

So if no other noise sources are taken into account and one integrates the signal over some time period (Δt), such that $B = 1/2\Delta t$ and $R = N/\Delta t$, one just gets counting statistics:

$$\delta A = \frac{\sigma_{I_k}}{I_k} = \frac{1}{\sqrt{N}} . \tag{1.5}$$

So far, from the point of view of noise, the total number of photoelectrons generated at the cathode doesn't seem to matter, since it cancels out in eqn. 1.4. Of course this isn't really true, as I will show below. Aside from actually making a noise contribution, a reasonably large number of photoelectrons is needed, since the main physics signal is competing with the background signals, as well as the PMT dark current. So increasing $\langle n \rangle$ for the physics signal, relative to $\langle n \rangle$ for the background signals and dark current is of major importance.

Accounting for additional (excess) noise sources modifies eqn. 1.5 to

$$\delta A = \frac{\sigma_{I_k}}{I_k} = \frac{\sqrt{1 + \alpha^2}}{\sqrt{N}} \tag{1.6}$$

and $\alpha^2 = \alpha^2_{DR} + \alpha^2_{PMT} + \alpha^2_{E}$ incorporates noise components stemming from the finite detector resolution (RD), PMT gain noise (PMT), and electronics noise (E). The first two terms are discussed in this document. The electronics noise is primarily thermal/resistive (Johnson noise $\alpha^2_E = 4k_B RTB$, with Boltzmann constant k_B and temperature T in Kelvin) which is addressed with appropriate electronic design. This is discussed in a separate document on electronics design. One of the challenges in the detector design is to keep the the excess noise as small as possible. It turns out that it is easy to inadvertently increase the excess noise from poor detector resolution or poor PMT design to levels that could increase the running time by as much as 20% or more.

The detector signal at the PMT output is a result of a series of cascaded processes, each of which is sampling a separate probability distribution (although they are often of the same type, e.g. Poisson), starting with the generation and transmission of photons (e.g. in a Cherenkov detector), through the photoelectric effect at the cathode, and ending with the dynode gain cascade. I will briefly review the method of probability generating functions and cascading probabilities, before moving on to calculate the excess noise factors.

2 Probability Generating Functions and Cascaded Processes

This section is a short summary of the relevant information found in references [1, 2, 4, 5]. A cascading process is one in which the output of one process serves as the input to another. This can be continued indefinitely, until there are no other processes to evaluate. If the underlying probability distributions are all known, the outcome of the overall process can be evaluated mathematically by progressive substitution of the distributions and then working out the analytic expression. However, even if the distributions are all known (which in general they are not), this can quickly become rather involved, for even a modest number of cascades. The method of probability generating functions, on the other hand, allows one to get expressions in terms of variances and means (the quantities we measure), without knowing (or even needing) the exact underlying distribution, and with relative ease.

2.1 The Probability Generating Function

A suitable generating function for an arbitrary discrete probability distribution $p(x)$ is

$$G_p(u) = p(0)u^0 + p(1)u^1 + p(2)u^2 + ... + p(x)u^x + ... = \sum_{n=0}^{\infty} p(n)u^n \ . \tag{2.1}$$

Taking the partial derivative w.r.t. u gives

$$G_p'(u) = \sum_{n=0}^{\infty} np(n)u^{n-1} \ .$$

Setting $u = 1$, then leads to the definition of the sample mean for the distribution $p(x)$

$$m_p \equiv G_p'(1) = \sum_{n=0}^{\infty} np(n) = \langle n \rangle \ . \tag{2.2}$$

The second derivative gives

$$G_p''(1) = \sum_{n=0}^{\infty} n(n-1)p(n) = \langle n(n-1) \rangle \ . \tag{2.3}$$

We can combine the results and extract the sample variance as follows:

$$\begin{aligned} G_p''(1) + G_p'(1) - (G_p'(1))^2 &= \langle n(n-1) \rangle + \langle n \rangle - (\langle n \rangle)^2 \\ &= \langle n^2 \rangle - \langle n \rangle + \langle n \rangle - (\langle n \rangle)^2 \\ &= \langle n^2 \rangle - (\langle n \rangle)^2 \ . \end{aligned} \tag{2.4}$$

So the variance is related to the partial derivatives of the generating function by

$$\sigma_p^2 \equiv G_p^{''}(1) + G_p^{'}(1) - (G_p^{'}(1))^2 \ . \tag{2.5}$$

2.2 Cascaded Processes

If the number of samples taken from a randomly distributed population is itself a random variable, associated with some probability distribution, then the two probability distributions form a cascade. This can be extended to any number of successive processes in a chain of events that are selected according to a set of underlying probability distributions for each process. The relevant example for the integrating detectors involves a set of N detector events, each producing p_i photons ($i = 1..N$), each of which produce m_j photoelectrons ($j = 1..p_i$), and continuing on in this fashion for each of the dynodes that are used in the PMT.

In terms of the generating functions the combined, cascaded process is expressed as [2]

$$G_{abc..z} = G_a(G_b(G_c(...(G_z(u))))) \ . \tag{2.6}$$

This is a recursion formula, where the processes take place in order, starting with a and ending with z. Introductions to the use of generating functions can be found, for example, in [4] and an easily accessible (and nicely explained) proof of eqn. 2.6 can be found at [5] (theorem 6.3, page 124 - although there it is used in the context of branching processes in population growth).

Taking the first and second derivatives in eqn. 2.6 and using eqns. 2.3 and 2.5, we can calculate the corresponding variance, mean, and relative error of the combined process. I will do this explicitly for the case of detector resolution below, taking into account the the three processes mentioned above. I then extend this to include dynode processes, by analogy (sparing myself and the reader the copious manipulations - but this is discussed in much more detail in [2]).

For three independent consecutive random processes, the total generating function is

$$G_{abc}(u) = G_a(G_b(G_c(u))) \ . \tag{2.7}$$

Taking the partial derivatives with respect to u gives

$$G_{abc}^{'}(u) = G_a^{'}(G_b(G_c(u)))G_b^{'}(G_c(u))G_c^{'}(u)$$

and

$$\begin{aligned} G_{abc}^{''}(u) &= G_a^{''}(G_b(G_c(u)))(G_b^{'}(G_c(u))G_c^{'}(u))^2 \\ &+ G_a^{'}(G_b(G_c(u)))G_b^{''}(G_c(u))(G_c^{'}(u))^2 \\ &+ G_a^{'}(G_b(G_c(u)))G_b^{'}(G_c(u))G_c^{''}(u) \ . \end{aligned}$$

So using $G_w(1) = 1$, $G_w^{'}(1) = m_w$ (the mean), and $G_w^{''}(1) = \sigma_w^2 + m_w^2 - m_w$ (the variance), from eqns. 2.3 and 2.5, we find that

$$G_{abc}^{'}(1) = G_a^{'}(1)G_b^{'}(1)G_c^{'}(1) = m_a m_b m_c \equiv m_{abc} \ . \tag{2.8}$$

Where m_a, m_b, and m_c are the mean values of the individual processes. From eqn. 2.10, we also find that

$$\begin{aligned} G''_{abc}(1) &= G''_a(1)(G'_b(1)G'_c(1))^2 + G'_a(1)G''_b(1)(G'_c(1))^2 + G'_a(1)G'_b(1)G''_c(1) \\ &= \left(\sigma_a^2 + m_a^2 - m_a\right) m_b^2 m_c^2 + \\ &\quad m_a \left(\sigma_b^2 + m_b^2 - m_b\right) m_c^2 + \\ &\quad m_a m_b \left(\sigma_c^2 + m_c^2 - m_c\right) \ . \end{aligned} \tag{2.9}$$

Combining eqns. 2.8 and 2.9, we then find for the total variance

$$\begin{aligned} \sigma_{abc}^2 &= G''_{abc}(1) + G'_{abc}(1) - (G'_{abc}(1))^2 \\ &= \sigma_a^2 m_b^2 m_c^2 + \\ &\quad \sigma_b^2 m_a m_c^2 + \\ &\quad \sigma_c^2 m_a m_b \ . \end{aligned} \tag{2.10}$$

Finally, using this and the result for the combined mean (eqn. 2.8), we can calculate the relative variance for the three stage process with arbitrary discrete probability distributions:

$$\frac{\sigma_{abc}^2}{m_{abc}^2} = \frac{\sigma_a^2}{m_a^2} + \frac{\sigma_b^2}{m_a m_b^2} + \frac{\sigma_c^2}{m_a m_b m_c^2} \ . \tag{2.11}$$

3 Excess Noise due to Detector Resolution

In applying eqn. 2.11 to the detector resolution, the process labeled above as a, b, c represent the following:

m_a is the mean number of events in the detector in some time interval

m_b is the mean number of photons per event

m_c is the mean number of photo-electrons per photon

The corresponding variances are, of course, σ_a^2, σ_b^2, and σ_c^2. We can factor out the common factor m_a in the RHS denominator of eqn. 2.11 and combine the last two terms, to get

$$\frac{\sigma_{abc}^2}{m_{abc}^2} = \frac{1}{m_a}\left(\frac{\sigma_a^2}{m_a} + \frac{m_c^2\sigma_b^2 + m_b\sigma_c^2}{m_c^2 m_b^2}\right) \ . \tag{3.1}$$

The mean values for m_b and m_c are not usually measured separately. Instead, one usually measures the combination $\langle n_{pe}\rangle = m_b m_c$, which is the mean number of photoelectrons per detector event, and the corresponding variance $\sigma_{n_{pe}}^2 = m_c^2\sigma_b^2 + m_b\sigma_c^2$. Likewise, the PMT cathode current has mean $I_k = m_a m_b m_c$, and variance $\sigma_{abc}^2 = \sigma_k^2$. With all of that, we obtain

$$\frac{\sigma_{I_k}^2}{I_k^2} = \frac{1}{m_a}\left(\frac{\sigma_a^2}{m_a} + \frac{\sigma_{n_{pe}}^2}{\langle n_{pe}\rangle^2}\right) \ . \tag{3.2}$$

Integration of the signal over a time period Δt (which is equivalent to multiplying by the bandwidth set by the integration period), we get $N = m_a \Delta t$ so that

$$\frac{\sigma_{I_k}}{I_k} = \frac{1}{\sqrt{N}} \sqrt{\left(\frac{\sigma_a^2}{m_a} + \frac{\sigma_{n_{pe}}^2}{\langle n_{pe} \rangle^2} \right)} \ . \tag{3.3}$$

Up to now, no assumptions have been made about the underlying probability distributions, so this is a general result. The first term under the square-root depends primarily on beam and target properties and the corresponding sample distribution would generally not be simple or well defined. However, if the sample period is short in comparison to target variations, beam drifts, and other non-random variations, the variance in the average event rate may follow a Poisson distribution, so that $\sigma_a^2 = N/\Delta t$ and

$$\frac{\sigma_{I_k}}{I_k} = \frac{1}{\sqrt{N}} \sqrt{\left(1 + \frac{\sigma_{n_{pe}}^2}{\langle n_{pe} \rangle^2} \right)} \ . \tag{3.4}$$

The excess noise component due to finite detector resolution is $\sigma_{n_{pe}}^2 / \langle n_{pe} \rangle^2$. It can be minimized by carefully optimizing the physical design of the detector active components (quartz and light guides) and by choosing a high quantum efficiency PMT at the right wavelength. Generally speaking, larger number of photoelectron count is better. Ideally, the distribution in the number of photoelectrons would be Poissonian, such that $\sigma_{n_{pe}}^2 / \langle n_{pe} \rangle^2 \simeq 1/\langle n_{pe} \rangle$, but this is not usually the case, since this component also depends on shower activity in the detector and light transmission/reflection processes. We therefore measure the detector resolution and optimize the prototype geometry to reduce the level of excess noise to an acceptable level.

4 Excess Noise due to PMT Gain

The PMT gain is facilitated by successive emission of additional electrons at each of the dynodes (those that are used), for each incident electron. Since the number of electrons that hit a given dynode is itself a random variable, each dynode process again just forms another step in the overall cascade. So one can expand the calculation in sec. 2.2 to include the noise contribution from the dynode stages. To illustrate how this influences the excess noise component, we can start with one dynode stage.

If we include one more cascade, eqn. 2.7 just changes to

$$G_{abcd}(u) = G_a(G_b(G_c(G_d(u)))) \ . \tag{4.1}$$

Then, calculating the first and second derivatives and combining results gives:

$$\frac{\sigma_{abcd}^2}{m_{abcd}^2} = \frac{\sigma_a^2}{m_a^2} + \frac{\sigma_b^2}{m_a m_b^2} + \frac{\sigma_c^2}{m_a m_b m_c^2} + \frac{\sigma_d^2}{m_a m_b m_c m_d^2} \ . \tag{4.2}$$

The new factor $m_d \equiv \delta_{d1}$ is now the gain at the first dynode, producing a mean dynode current of $I_{d1} = m_a m_b m_c m_d = (N/\Delta t) \langle n_{pe} \rangle \delta_{d1}$. So, with the same quantities defined before, one can rewrite eqn. 4.2 as

$$\frac{\sigma_{d1}}{I_{d1}} = \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{\sigma^2_{\delta_{d1}}}{\langle n_{pe}\rangle \delta^2_{d1}}\right)} \ . \tag{4.3}$$

The creation of secondary electrons at the dynodes follows a Poisson distribution so that $\sigma^2_{\delta_{d1}} = \delta_{d1}$ and

$$\frac{\sigma_{d1}}{I_{d1}} = \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{1}{\langle n_{pe}\rangle \delta_{d1}}\right)} \ . \tag{4.4}$$

At this point a clear pattern emerges and the effect of the next dynode can be inferred:

$$\frac{\sigma^2_{abcde}}{m^2_{abcde}} = \frac{\sigma^2_a}{m^2_a} + \frac{\sigma^2_b}{m_a m^2_b} + \frac{\sigma^2_c}{m_a m_b m^2_c} + \frac{\sigma^2_d}{m_a m_b m_c m^2_d} + \frac{\sigma^2_e}{m_a m_b m_c m_d m^2_e} \ . \tag{4.5}$$

Where $m_e \equiv \delta_{d2}$ is the gain at the second dynode and $\sigma^2_e = \delta_{d2}$ is the corresponding variance.

$$\frac{\sigma_{d2}}{I_{d2}} = \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{1}{\langle n_{pe}\rangle \delta_{d1}}\left(1 + \frac{1}{\delta_{d2}}\right)\right)} \ . \tag{4.6}$$

The next stage after that would be:

$$\frac{\sigma_{d3}}{I_{d3}} = \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{1}{\langle n_{pe}\rangle \delta_{d1}}\left(1 + \frac{1}{\delta_{d2}}\left(1 + \frac{1}{\delta_{d3}}\right)\right)\right)} \ . \tag{4.7}$$

For a PMT with n gain stages, we would then have the following anode output:

$$\frac{\sigma_{a}}{I_{a}} = \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{1}{\langle n_{pe}\rangle \delta_{d1}}\left(1 + \frac{1}{\delta_{d2}} + \frac{1}{\delta_{d2}\delta_{d3}} + \cdots + \frac{1}{\delta_{d2}\delta_{d3}\cdots\delta_{dn}}\right)\right)} \ . \tag{4.8}$$

If n is sufficiently large and all gains beyond the first dynode are the same (δ), then eqn 4.8 is approximately equal to

$$\frac{\sigma_{a}}{I_{a}} \simeq \frac{1}{\sqrt{N}}\sqrt{\left(1 + \frac{\sigma^2_{n_{pe}}}{\langle n_{pe}\rangle^2} + \frac{1}{\langle n_{pe}\rangle \delta_{d1}}\left(\frac{\delta}{\delta - 1}\right)\right)} \ . \tag{4.9}$$

This approximation is reasonably good for high gain PMTs, where most dynodes of the PMT are used. However, as discussed in the next section, for the MOLLER/P2 integration mode measurements, we need to keep the gain relatively low and will need to be restricted to only use the first couple of dynodes, to reduce the gain noise and maintain reasonable linearity. In that case, the explicit expression given by eqn. 4.8, is more appropriate.

5 Minimizing PMT Gain Noise under Stability and Linearity Constraint

For P2 and MOLLER, the high rate, integrating detectors produce cathode currents between 10 and 25 nA, depending on the actual rate and the number of photoelectrons per event. The cathode and anode current limits are specified by the PMT manufacturer and the maximum anode current sets the maximum gain that can be applied, given a certain cathode current. The PMT that is currently under evaluation for both MOLLER and P2 is the 9305QKB PMT by ET Enterprises. It has a maximum anode current of 100 μA and one usually wants to stay well below that, to ensure longevity and relative long term gain stability of the PMT. The manufacturer of the 9305QKB does not specify a maximum number of Coulombs that can be drawn from the anode, but other sources [7] give a definition of PMT *end-of-life* when the anode sensitivity has dropped by a factor 2, which typically happens somewhere between 300 and 1000 Coulombs [7].

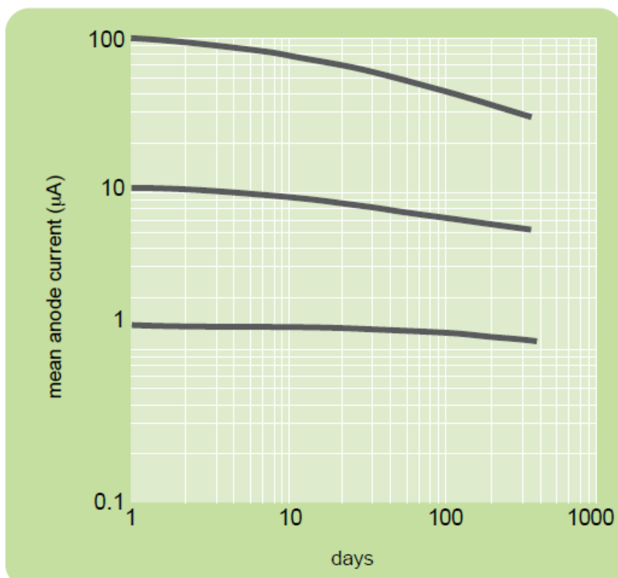

Figure 1: Plot of anode current stability under continuous illumination and constant bias voltage, taken over one year. The higher the anode current, the more pronounced is the gain drop over time. The figure was taken from [6].

Figure 1 shows the long term gain stability, measured by ET Enterprises (not necessarily with the 9305QKB PMT) for a few different anode currents [6]. The figure shows that, for a starting gain that produces a continuous anode current of 10 μA at the beginning, the PMT will be at its nominal *end-of-life* state after a little less than a year of running. This is roughly consistent with 300 Coulombs of charge drawn over the year. MOLLER will run for 344 full days (or 8256 hours) and P2 will run for 10000 hours or about 416 full days. In current mode, the reduced gain can be countered by increasing the overall bias voltage over time, but as I will show below, we want to start off with a relatively high bias voltage and low number of stages to begin with and the PMT maximum voltage ratings limit the possibility for increasing the bias voltage later on. For these reasons it is prudent to keep the anode current at or below 10 μA, which means that the total PMT gain should be no more than around 1000, depending on the exact detector rate and photoelectron efficiency. Meanwhile, eqn. 4.8 shows that large dynode gain and a high photoelectron efficiency reduce excess noise. The list below summarizes the various, partially competing constraints.

1. Optimize the detector resolution, which implies maximizing n_{pe}, as long as this is not achieved while introducing shower noise or other non-Poissonian (non-cathode related) noise sources.

2. Maximize the inter-dynode gain (δ) for, at least, the first and second stages.

3. Maintain a gain that is low enough to keep the anode current at or below 10 μA.

4. Maintain reasonably linear operating conditions, which means keeping the inter-dynode and dynode to anode voltage reasonably high.

The MOLLER/P2 candidate PMT has high gain SbCs dynodes for which the gain at the nth dynode satisfies the following relation [2, 3]

$$\delta_{dn} = 0.172\,(V_{dn-1,dn})^{0.72} \quad . \tag{5.1}$$

Where $V_{n-1,n}$ is the voltage drop between the nth dynode and the one before. So choosing a large inter-dynode voltage increases the dynode gain, which decreases the noise. As seen in eqn. 4.8, having a large gain at the first dynode is critical, to reduce the noise. We also know that a low cathode to first dynode voltage (and a low or unstable voltage between the last dynode and the anode) leads to increased non-linear PMT behavior. At the same time, however, we are limited in how high we can make it and still keep the overall gain low enough. So for integration mode running, we want to use relatively few dynodes with relatively high gain. For the candidate 9305QKB PMT, the maximum cathode to first dynode voltage is $V_{k,d1} = 450$ V and the maximum inter-dynode voltage is $V_{dn-1,dn} = 300$ V. If all of the inter-dynode voltages are the same ($V_{d,d}$), then the total gain for an n stage voltage divider is [2]

$$G = aV_{k,d1}^{b}\left(aV_{d,d}^{b}\right)^{(n-1)} \quad . \tag{5.2}$$

Where $a = 0.172$ and $b = 0.72$, for the SbCs dynode. We can write the total bias voltage as $V_{PMT} = mv + nv$ (an n-stage PMT has $n+1$ voltage divisions), where $V_{k,d1} = mv$ and $V_{d,d} = v$. With that

$$G = a\left(\frac{mV}{m+n}\right)^{b}\left(a\left(\frac{V_{PMT}}{m+n}\right)^{b}\right)^{(n-1)} = a^{n}m^{b}\left(\frac{V_{PMT}}{m+n}\right)^{nb} \quad . \tag{5.3}$$

Calculating $\partial G/\partial m = 0$, one finds that G is maximized when $m = n/(n-1)$, for a fixed bias voltage V_{PMT}.

5.1 Example: 3 Stage PMT Base

As an example, consider a voltage divider with three gain stages (3 dynodes and 4 voltage drops). We then maximize the gain if $m = 3/2$, which gives the optimal combination for $G = \delta_{d1}\delta_{d2}\delta_{d3}$. If we want an overall gain of 1000, then, from eqn. 5.3, the PMT bias voltage is

$$1000 = (0.172)^3 \left(\frac{3}{2}\right)^{0.72} \left(\frac{2V_{PMT}}{9}\right)^{3\times 0.72} \quad \Rightarrow \quad V_{PMT} \simeq 1110 \text{ V} .$$

Figure 2 shows the corresponding gain curve, as a function relative cathode to first dynode voltage $V_{k,d1}/V_{d,d}$.

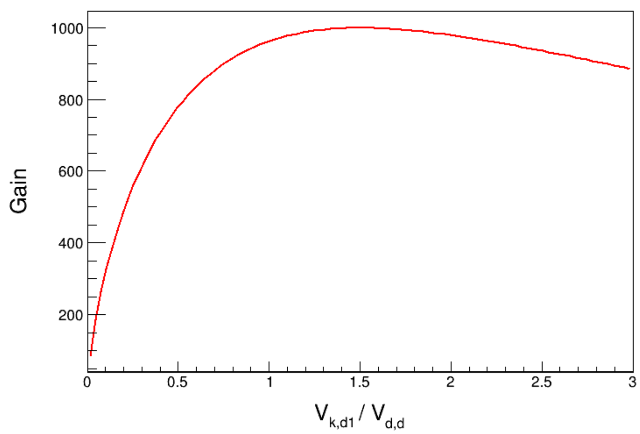

Figure 2: Gain dependence on the fractional relationship between $V_{k,d1}$ and $V_{d,d}$, for a 3 stage voltage divider, for a bias voltage of $\simeq$ 1110 V.

So with this we find $v = 2V_{PMT}/9 = V_{d,d} = 247$ V and $V_{k,d1} = mv = 370$ V. Substituting these into eqn. 5.1 then gives

$$\begin{aligned} \delta_{d1} &= 0.172(370)^{0.72} \simeq 12 \\ \delta_{d} &= 0.172(247)^{0.72} \simeq 9 . \end{aligned}$$

Where $\delta_d 1$ is the gain from the first dynode and δ_d is the gain from from all other dynodes. The total gain is about $G = 12 \times 9 \times 9 = 972$.

WE can now go back and use eqn. 4.8, to calculate the theoretically expected noise contribution from such a PMT base. The PMT noise contribution is given by

$$\alpha^2 = \frac{1}{\langle n_{pe}\rangle \delta_{d1}} \left(1 + \frac{1}{\delta_{d2}} + \frac{1}{\delta_{d2}\delta_{d3}}\right) . \tag{5.4}$$

From measurement, the MOLLER ring 5 detectors get about 30 photoelectrons, so for those detectors

$$\alpha^2_{PMT} = \frac{1}{30 \times 12} \left(1 + \frac{1}{9} + \frac{1}{9 \times 9}\right) \simeq 0.003 . \tag{5.5}$$

For P2 the detectors get about 50 photoelectrons [8], which gives

$$\alpha^2_{PMT} = \frac{1}{50 \times 12}\left(1 + \frac{1}{9} + \frac{1}{9 \times 9}\right) \simeq 0.002 \; . \tag{5.6}$$

In both cases, this is much less than the desired contribution from detector resolution alone.

By contrast, suppose one were to use a 7 stage base at a gain of 1000. By eqns. 5.3 and 5.1 one finds $\delta_d = 2.64$ and $\delta_d 1 = 2.95$. With these, the noise factor for this PMT base design would be $\alpha^2_{PMT} \simeq 0.018$ for MOLLER and $\alpha^2_{PMT} \simeq 0.011$ for P2. These values are much closer to the desired total excess noise values for both experiments. This is aside from the fact that the inter-dynode voltages are very low for such a design ($V_{d,d} \simeq 44$ V), while $V_{k,d1} \simeq 51$ V. These are much too low and would likely produce a highly non-linear performance.

6 Summary

The careful evaluation of all sources of random noise in the detector signal chain shows that the design of the physical detector components (quartz and light guide) plays a crucial role in the reduction of excess noise. In addition, the number of photoelectrons forms an important factor in suppressing the excess noise contribution from dynode gain fluctuations. The gain must be kept low, distributed across a small number of dynodes, to minimize the noise, ensure longevity, and maintain good linearity.

References

[1] W. Davenport and W. Root, "Intrduction to the Theory of Random Signals and Noise" , Wiley-IEEE Press , 1958 and 1987

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